

Asymptotic generalized bivariate extreme with random index

M. A. Abd Elgawad^{a,b}, A. M. Elsayah^{a,c,d,*}, Hong Qin^a and Ting Yan^a

^a *Faculty of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China*

^b *Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt*

^c *Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt*

^d *Division of Science and Technology, BNU-HKBU United International College, Zhuhai 519085, China*

Abstract

In many biological, agricultural, military activity problems and in some quality control problems, it is almost impossible to have a fixed sample size, because some observations are always lost for various reasons. Therefore, the sample size itself is considered frequently to be a random variable (rv). The class of limit distribution functions (df's) of the random bivariate extreme generalized order statistics (GOS) from independent and identically distributed rv's are fully characterized. When the random sample size is assumed to be independent of the basic variables and its df is assumed to converge weakly to a non-degenerate limit, the necessary and sufficient conditions for the weak convergence of the random bivariate extreme GOS are obtained. Furthermore, when the interrelation of the random size and the basic rv's is not restricted, sufficient conditions of the convergence and the forms of the limit df's are deduced. Illustrative examples are given which lend further support to our theoretical results.

Keywords: Weak convergence; Random sample size; Generalized order statistics; Generalized bivariate extreme.

1 Introduction

The concept of generalized order statistics (GOS) have been introduced by Kamps (1995). It's enable a unified approach to ascendingly ordered random variables (rv's) as ordinary order statistics (oos), sequential order statistics (sos), order statistics with non integral sample size, progressively type II censored order statistics (pos), record values, k th record

*Corresponding author. E-mail: amelsawah@uic.edu.hk, a.elsawah@zu.edu.eg, a_elsawah85@yahoo.com

values and Pfeifer's records. Let $\gamma_n = k > 0$, $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0$, $r = 1, 2, \dots, n-1$, and $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathfrak{R}^{n-1}$. Then the rv's $X_{r:n} \equiv X(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$, are called GOS based on the distribution function (df) F with density function f which are defined by their probability density function (pdf)

$$f_{1,2,\dots,n:n}^{(\tilde{m},k)}(x_1, x_2, \dots, x_n) = \left(\prod_{j=1}^n \gamma_j \right) \left(\prod_{j=1}^{n-1} (1 - F(x_j))^{\gamma_j - \gamma_{j+1} - 1} f(x_j) \right) \\ \times (1 - F(x_n))^{\gamma_n - 1} f(x_n),$$

where $F^{-1}(0) \leq x_1 \leq \dots \leq x_n \leq F^{-1}(1)$.

In this work, we consider a wide subclass of GOS, by assuming $\gamma_j - \gamma_{j+1} = m + 1 > 0$. This subclass is known as m -GOS. Clearly many important practical models of m -GOS are included such as oos, order statistics with non integer sample size and sos. The marginal df's of the r th and \hat{r} th m -GOS (Nasri- Roudsari, 1996 and Barakat, 2007) are represented by $\Phi_{r:n}^{(m,k)}(x) = I_{L_m(x)}(r, N - r + 1)$ and $\Phi_{\hat{r}:n}^{(m,k)}(x) = I_{L_m(x)}(N - R_r + 1, R_r)$, respectively, where $\hat{r} = n - r + 1$, $L_m(x) = 1 - (1 - F(x))^{m+1}$, $I_x(a, b) = \frac{1}{\beta(a, b)} \int_0^x t^{a-1} (1 - t)^{b-1} dt$ denotes the incomplete beta ratio function, $N = \ell + n - 1$, $R_r = \ell + r - 1$ and $\ell = \frac{k}{m+1}$. Moreover, by using the results of Kamps (1995), we can write explicitly the joint df's of the r th and s th m -GOS, $m \neq -1$, $1 \leq r < s \leq n$, as:

$$\Phi_{r,s:n}^{(m,k)}(x, y) = C_n^* \int_0^{F(x)} \int_{\xi}^{F(y)} \bar{\xi}^m \bar{\eta}^{\gamma_s - 1} (1 - \bar{\xi}^{m+1})^{r-1} (\bar{\xi}^{m+1} - \bar{\eta}^{m+1})^{s-r-1} d\eta d\xi, \quad x \leq y, \quad (1.1)$$

where $C_n^* = \frac{(m+1)^2 \Gamma(N+1)}{\Gamma(N-s+1)(r-1)!(s-r-1)!}$ and $\Gamma(\cdot)$ is the usual gamma function. Recently Barakat et al. (2014a) studied the limit df's of joint extreme m -GOS, for a fixed sample size. Moreover, the asymptotic behavior for bivariate df of the lower-lower (l-l), upper-upper (u-u) and lower-upper (l-u) extreme m -GOS in Barakat et al. (2014b).

In the last few years much efforts had been devoted to investigate the limit df's of independent rv's with random sample size. The appearance of this trend is naturally because many applications require the consideration of such problem. For example, in many biological, agricultural and in some quality control problems, it is almost impossible to have a fixed sample size because some observations always get lost for various reasons. Therefore, the sample size n itself is considered frequently to be a rv ν_n , where ν_n is independent of the basic variables (i.e., the original random sample) or in some applications the interrelation of the basic variables and the random sample size is not restricted. Limit theorems for extremes with random sample size indexes have been thoroughly studied in the above mentioned two particular cases :

1. The basic variables and sample size index are independents (see, Barakat, 1997).
2. The interrelation of the basic variables and the random sample size is not restricted (see, Barakat and El Shandidy, 1990, Barakat, 1997 and Barakat et al., 2015a).

Our aim in this paper is to characterize the asymptotic behavior of the bivariate df's of the (l-l), (u-u) and (l-u) extreme m -GOS with random sample size. When the random sample size is assumed to be independent of the basic variables and its df is assumed to converge weakly to a non-degenerate limit, the necessary and sufficient conditions for the weak convergence of the random bivariate extreme m -GOS are obtained. Furthermore, when the interrelation of the random size and the basic rv's is not restricted, sufficient conditions of the convergence and the forms of the limit df's are deduced. An illustrative examples are given which lend further support to our theoretical results. Throughout this paper the convergence in probability and the weak convergence, as $n \rightarrow \infty$, respectively, denoted as " \xrightarrow{p} " and " \xrightarrow{w} ".

2 Asymptotic random bivariate extreme under m -GOS

2.1 Random sample size and basic rv's are independents

In this subsection we deal with the weak convergence of bivariate df's of the (u-u), (l-l) and (l-u) extreme m -GOS are fully characterized in Theorems 2.1, 2.2 and 2.3, respectively. When the sample size itself is a rv ν_n , which is assumed to be independent of the basic variables $X_{r:n}, r = 1, 2, \dots, n$.

Theorem 2.1. Consider the following three conditions :

$$\Phi_{\hat{r}, \hat{s}:n}^{(m,k)}(x_n, y_n) = P(Z_{\hat{r}, \hat{s}:n}^{(n)} < \mathbf{x}) = P(Z_{\hat{r}:n}^{(n)} < x, Z_{\hat{s}:n}^{(n)} < y) \xrightarrow{w} \hat{\Phi}_{r,s}^{(m,k)}(x, y), x \leq y, \quad (i)$$

$$H_n(nz) = P\left(\frac{\nu_n}{n} < z\right) \xrightarrow{w} H(z), \quad (ii)$$

$$\begin{aligned} \Phi_{\hat{r}, \hat{s}:n}^{(m,k)}(x_n, y_n) &= P(Z_{\hat{r}, \hat{s}:n}^{(n)} < \mathbf{x}) = P(Z_{\hat{r}:n}^{(n)} < x, Z_{\hat{s}:n}^{(n)} < y) \xrightarrow{w} \hat{\Psi}_{r,s}^{(m,k)}(x, y) \\ &= \int_0^\infty \bar{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH(z). \end{aligned} \quad (iii)$$

Then any two of the above conditions imply the remaining one, where $x_n = x_{1n} = a_n x_1 + b_n, y_n = x_{2n} = a_n x_2 + b_n, a_n > 0, b_n$ are suitable normalizing constants, $\mathbf{x} = (x_1, x_2) = (x, y), \hat{r} = n - r + 1 < n - s + 1 = \hat{s}, Z_{\hat{r}:n}^{(n)} = \frac{X_{\hat{r}:n} - b_n}{a_n}, i = 1, 2, (\hat{r}_1, \hat{r}_2) = (\hat{r}, \hat{s}), \hat{\Phi}_{r,s}^{(m,k)}(x, y)$

is a non-degenerate df, $H(z)$ is a df with $H(+0) = 0$,

$$\overline{\Omega}_{r,s}^{(m,k)}(\kappa_1, \kappa_2) = \begin{cases} 1 - \Gamma_{R_s}(\kappa_2^{m+1}), & x \geq y, \\ 1 - \Gamma_{R_r}(\kappa_1^{m+1}) - \frac{1}{\Gamma(R_r)} \int_{\kappa_1^{m+1}}^{\infty} I\left(\frac{\kappa_2^{m+1}}{u}\right) (R_s, R_r - R_s) u^{R_r-1} e^{-u} du, & x \leq y, \end{cases}$$

$\Gamma_r(x) = \frac{1}{\Gamma(r)} \int_0^x \theta^{r-1} e^{-\theta} d\theta$ denotes the incomplete gamma ratio function, $\kappa_i = \mathcal{U}_{j;\alpha}(x_i)$, $i = 1, 2, j \in \{1, 2, 3\}$, $\mathcal{U}_{1;\alpha}(x_i) = x_i^{-\alpha}$, $x_i \leq 0$; $\mathcal{U}_{2;\alpha}(x_i) = (-x_i)^\alpha$, $x_i > 0$, $\alpha > 0$ and $\mathcal{U}_3(x_i) = \mathcal{U}_{3;\alpha}(x_i) = e^{-x_i}$, $\forall x_i$.

Remark 2.1. The continuity of the limit df $\hat{\Phi}_{r,s}^{(m,k)}(x, y)$ in (i) implies the continuity of the limit $\hat{\Psi}_{r,s}^{(m,k)}(x, y)$. Hence the convergence in (iii) is uniform with respect to x and y .

Remark 2.2. It is natural to look for the limitations on ν_n , under which we get the relation $\hat{\Phi}_{r,s}^{(m,k)}(x, y) \equiv \hat{\Psi}_{r,s}^{(m,k)}(x, y) \forall x, y$. In view of Theorem 2.1, the last equation is satisfied if and only if the df $H(z)$ is degenerate at one, which means the asymptotically almost randomness of ν_n . We assume, due to Remark 2.2, that $H(z)$ is a non-degenerate df and $H(+0) = 0$, i.e., continuous at zero.

Proof of the implication (i) + (ii) \Rightarrow (iii): First, we note that $\Phi_{\hat{r}, \hat{s}; n}^{(m,k)}(x_n, y_n)$ can be written in the form (see, Theorem 2.3 in Barakat et al., 2014b)

$$\Phi_{\hat{r}, \hat{s}; n}^{(m,k)}(x_n, y_n) = 1 - \Gamma_{R_r}(N \overline{L}_m(x_n)) - \frac{1}{\Gamma(R_r)} \int_{N \overline{L}_m(x_n)}^N I_{\frac{N \overline{L}_m(y_n)}{u}}(R_s, R_r - R_s) u^{R_r-1} e^{-u} du, \quad (2.1)$$

where $\overline{L}_m(\cdot) = 1 - L_m(\cdot)$. Now by using the total probability rule we get,

$$\Phi_{\hat{r}, \hat{s}; \nu_n}^{(m,k)}(x_n, y_n) = \sum_{t=r}^{\infty} \Phi_{\hat{r}, \hat{s}; t}^{(m,k)}(x_n, y_n) P(\nu_n = t). \quad (2.2)$$

Assume that $H_n(z) = \sum_{t \leq z} P(\nu_n = t) = P(\nu_n \leq z)$ and $z = [\frac{t}{n}]$, where $[\theta]$ denotes the greatest integer part of θ . Thus, the relation (2.1) show that the sum term in (2.2) is a Riemann sum of the integral

$$\Phi_{\hat{r}, \hat{s}; \nu_n}^{(m,k)}(x_n, y_n) = \int_0^{\infty} \Phi_{\hat{r}, \hat{s}; n}^{(m,k)}(x_n, y_n, z) dH_n(nz), \quad (2.3)$$

where, for sufficiently large n , we have

$$\Phi_{\hat{r}, \hat{s}; n}^{(m,k)}(x_n, y_n, z) = 1 - \Gamma_{R_r}(z \hat{N} \overline{L}_m(x_n)) - \frac{1}{\Gamma(R_r)} \int_{z \hat{N} \overline{L}_m(x_n)}^{z \hat{N}} I_{\frac{z \hat{N} \overline{L}_m(y_n)}{u}}(R_s, R_r - R_s) u^{R_r-1} e^{-u} du,$$

where $\hat{N} = (\frac{\ell-1}{z} + n) \sim n$. Appealing to the condition (i), Theorem 2.3 in Barakat et al. (2014b) and Remark 2.2, we get

$$\Phi_{\hat{r}, \hat{s}; n}^{(m,k)}(x_n, y_n, z) \xrightarrow{w} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2), \quad (2.4)$$

where the convergence is uniform with respect to x and y , over any finite interval of z . Now, let ξ be a continuity point of $H(z)$ such that $1 - H(\xi) < \epsilon$, (ϵ is an arbitrary small value). Then, we have

$$\int_{\xi}^{\infty} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH(z) \leq \int_{\xi}^{\infty} dH(z) = 1 - H(\xi) < \epsilon. \quad (2.5)$$

In view of the condition (ii) we get, for sufficiently large n , that

$$\int_{\xi}^{\infty} \Phi_{\hat{r},\hat{s};n}^{(m,k)}(x_n, y_n, z) dH_n(nz) \leq 1 - H_n(n\xi) \leq 2(1 - H(\xi)) < 2\epsilon. \quad (2.6)$$

On the other hand, by the triangle inequality, we get

$$\begin{aligned} & \left| \int_0^{\xi} \Phi_{\hat{r},\hat{s};n}^{(m,k)}(x_n, y_n, z) dH_n(nz) - \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH(z) \right| \\ & \leq \left| \int_0^{\xi} \Phi_{\hat{r},\hat{s};n}^{(m,k)}(x_n, y_n, z) dH_n(nz) - \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH_n(nz) \right| \\ & \quad + \left| \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH_n(nz) - \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH(z) \right|, \end{aligned} \quad (2.7)$$

where the convergence in (2.4) is uniform over the finite interval $0 \leq z \leq \xi$. Therefore, for arbitrary $\epsilon > 0$ and for sufficiently large n , we have

$$\left| \int_0^{\xi} \left[\Phi_{\hat{r},\hat{s};n}^{(m,k)}(x_n, y_n, z) - \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) \right] dH_n(nz) \right| \leq \epsilon H_n(n\xi) \leq \epsilon. \quad (2.8)$$

In order to estimate the second difference on the right hand side of (2.7), we construct Riemann sums which are close to the integral there. Let T be a fixed number and $0 = \xi_0 < \xi_1 < \dots < \xi_T = \xi$ be continuity points of $H(z)$. Furthermore, let T and ξ_i be such that

$$\left| \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH_n(nz) - \sum_{i=1}^T \overline{\Omega}_{r,s}^{(m,k)}(\xi_i\kappa_1, \xi_i\kappa_2) (H_n(n\xi_i) - H_n(n\xi_{i-1})) \right| < \epsilon$$

and

$$\left| \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH(z) - \sum_{i=1}^T \overline{\Omega}_{r,s}^{(m,k)}(\xi_i\kappa_1, \xi_i\kappa_2) (H(\xi_i) - H(\xi_{i-1})) \right| < \epsilon.$$

Since, by the assumption $H_n(n\xi_i) \xrightarrow{\frac{w}{n}} H(\xi_i)$, $0 \leq i \leq T$, the two Riemann sums are closer to each other than ϵ for all n sufficiently large. Thus, once again by the triangle inequality, the absolute value of the difference of the integrals is smaller than 3ϵ . Combining this fact with (2.8), the left hand side of (2.7) becomes smaller than 4ϵ for all large n . Therefore, in view of (2.5), (2.6) and (2.4), we have

$$\left| \Phi_{\hat{r},\hat{s};\nu_n}^{(m,k)}(x_n, y_n) - \hat{\Psi}_{r,s}^{(m,k)}(x, y) \right| < \left| \int_0^{\xi} \Phi_{\hat{r},\hat{s};n}^{(m,k)}(x_n, y_n, z) dH_n(nz) - \int_0^{\xi} \overline{\Omega}_{r,s}^{(m,k)}(z\kappa_1, z\kappa_2) dH(z) \right|$$

$$+ \int_{\xi}^{\infty} \Phi_{\hat{r}, \hat{s}; n}^{(m, k)}(x_n, y_n, z) dH_n(nz) + \int_{\xi}^{\infty} \overline{\Omega}_{r, s}^{(m, k)}(z\kappa_1, z\kappa_2) dH(z) < 7\epsilon.$$

This completes the proof of the first part of the theorem.

Proof of the implication (i) + (iii) \Rightarrow (ii): Starting with the relation (2.3), we select a subsequence $\{n'\}$ of $\{n\}$ for which $H_{n'}(n'z)$ converges weakly to an extended df $H'(z)$ (i.e., $H'(\infty) - H'(0) \leq 1$ and such a subsequence exists by the compactness of df's). Then, by repeating the first part of the theorem for the subsequence $\{n'\}$, with the exception that we choose ξ so that $H'(\infty) - H'(\xi) < \epsilon$, we get $\hat{\Psi}_{r, s}^{(m, k)}(x, y) = \int_0^{\infty} \overline{\Omega}_{r, s}^{(m, k)}(z\kappa_1, z\kappa_2) dH'(z)$. Since the function $\hat{\Psi}_{r, s}^{(m, k)}(x, y)$ is a df, we get $\hat{\Psi}_{r, s}^{(m, k)}(\infty, \infty) = 1 = \int_0^{\infty} dH'(z) = H'(\infty) - H'(0)$, which implies that $H'(z)$ is a df. Now, if $H_n(nz)$ did not converge weakly, then we can select two subsequences $\{n'\}$ and $\{n''\}$ such that $H_{n'}(n'z) \xrightarrow[n']{w} H'(z)$ and $H_{n''}(n''z) \xrightarrow[n'']{w} H''(z)$, where $H'(z)$ and $H''(z)$ are df's. In this case, we get

$$\hat{\Psi}_{r, s}^{(m, k)}(x, y) = \int_0^{\infty} \overline{\Omega}_{r, s}^{(m, k)}(z\kappa_1, z\kappa_2) dH'(z) = \int_0^{\infty} \overline{\Omega}_{r, s}^{(m, k)}(z\kappa_1, z\kappa_2) dH''(z).$$

Thus, let $(y \rightarrow \infty)$, we get

$$\int_0^{\infty} \Gamma_{R_r}(z\kappa_1^{m+1}) dH'(z) = \int_0^{\infty} \Gamma_{R_r}(z\kappa_1^{m+1}) dH''(z). \quad (2.9)$$

Appealing to equation (2.9) and by using the same argument which is applied in the proof of the second part of Theorem 2.1 in Barakat, 1997, we can easily prove $H'(z) = H''(z)$.

This complete the proof of the second part.

Proof of the implication (ii) + (iii) \Rightarrow (i): For proving this part, we need first present the following lemma.

Lemma 2.1. For all $x_i, i = 1, 2$, we have

$$[1 - \Gamma_{R_{r_i}}(N\overline{L}_m(x_{in}))] - \sigma_{i, N} \leq P(Z_{\hat{r}_i; n}^{(n)} < x_i) \leq [1 - \Gamma_{R_{r_i}}(N\overline{L}_m(x_{in}))] + \rho_{i, N} \quad (2.10)$$

and

$$\Gamma_{R_{r_i}}(N\overline{L}_m(x_{in})) - \rho_{i, N} \leq P(Z_{\hat{r}_i; n}^{(n)} \geq x_i) \leq \Gamma_{R_{r_i}}(N\overline{L}_m(x_{in})) + \sigma_{i, N}, \quad (2.11)$$

where $0 < \rho_{i, N}, \sigma_{i, N} \xrightarrow[N]{\rightarrow} 0$ (or equivalently, as $n \rightarrow \infty$), $(R_{r_1}, R_{r_2}) = (R_r, R_s)$ and $(r_1, r_2) = (r, s)$.

Proof. Since $0 \leq \Gamma_{R_{r_i}}(x) \leq 1 \forall x$, the proof of the lemma will immediately follow from the result of Smirnov (1952) (Theorem 3, p. 133, or Lemma 2.1 in Barakat, 1997).

We now turn to the proof of the last part of Theorem 2.1. In view of Remark 2.1, we can assume, without any loss of generality, that the df $\hat{\Psi}_{r, s}^{(m, k)}(x, y)$ is continuous. Therefore,

the condition (iii) will be satisfied for all univariate marginals of $\hat{\Psi}_{r,s}^{(m,k)}(x, y)$, i.e., we have

$$\Phi_{\hat{r}_i:\nu_n}^{(m,k)}(x_{in}) \xrightarrow[n]{w} \hat{\Psi}_{r_i}^{(m,k)}(x_i), i = 1, 2, \quad (2.12)$$

where $\hat{\Psi}_{r_i}^{(m,k)}(x_i) = \int_0^\infty [1 - \Gamma_{R_{r_i}}(z\kappa_i^{m+1})]dH(z)$, $i = 1, 2$, is the marginals df's of $\hat{\Psi}_{r,s}^{(m,k)}(x, y)$.

We shall now prove

$$\Phi_{\hat{r}_i:n}^{(m,k)}(x_{in}) \xrightarrow[n]{w} \hat{\Phi}_{r_i}^{(m,k)}(x_i) = 1 - \Gamma_{R_{r_i}}(\kappa_i^{m+1}), i = 1, 2. \quad (2.13)$$

In view of Lemma 2.1, we first show that the sequence $\{Z_{\hat{r}_i:n}^{(n)}\}_n$, $i = 1, 2$, is stochastically bounded (see, Feller, 1979). If we assume the contrary, we would find $\varepsilon_{i,1}, \varepsilon_{i,2} > 0$ such that at least one of the two following relations

- (a) $\varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:n}^{(n)} \geq x_i) \geq \varepsilon_{i,1} > 0, \quad \forall x_i > 0, i = 1, 2,$
- (b) $\varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:n}^{(n)} < x_i) \geq \varepsilon_{i,2} > 0, \quad \forall x_i < 0, i = 1, 2$

is satisfied. The assertions (a) and (b) mean that the sequence $\{Z_{\hat{r}_i:n}^{(n)}\}_n$, is not stochastically bounded at the left $(-\infty)$ and at the right $(+\infty)$, respectively. Let the assumption (a) be true. Since $H(z)$ is non-degenerate df, we find $\varepsilon_0 > 0$ and $\beta > 0$ such that

$$P\left(\frac{\nu_n}{n} \geq \beta\right) \geq \varepsilon_0, \text{ for sufficiently large } n. \quad (2.14)$$

Using the following well known inequality, for $i = 1, 2$,

$$P\left(Z_{\hat{r}_i:n_1}^{(n)} \geq x_i\right) \geq P\left(Z_{\hat{r}_i:n_2}^{(n)} \geq x_i\right), \quad \forall n_1 \geq n_2. \quad (2.15)$$

We thus get the following inequalities, for sufficiently large n ,

$$\begin{aligned} P\left(Z_{\hat{r}_i:\nu_n}^{(n)} \geq x_i\right) &\geq \sum_{t \geq [n\beta]} P\left(Z_{\hat{r}_i:t}^{(n)} \geq x_i\right) P(\nu_n = t) \\ &\geq P\left(Z_{\hat{r}_i:[n\beta]}^{(n)} \geq x_i\right) P(\nu_n \geq [n\beta]) \geq \varepsilon_0 P\left(Z_{\hat{r}_i:[n\beta]}^{(n)} \geq x_i\right), \quad i = 1, 2, \end{aligned}$$

(note that $P(\nu_n \geq [n\beta]) \geq P(\nu_n \geq n\beta)$). Therefore,

$$\varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:\nu_n}^{(n)} \geq x_i) \geq \varepsilon_0 \varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:[n\beta]}^{(n)} \geq x_i).$$

Now, if we find $\varepsilon'_{i,1} > 0$ such that $\varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:[n\beta]}^{(n)} \geq x_i) \geq \varepsilon'_{i,1} > 0$, we get $\varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:\nu_n}^{(n)} \geq x_i) \geq \varepsilon_0 \varepsilon'_{i,1} > 0$, which contradicts the right stochastic boundedness of the sequence $\{Z_{\hat{r}_i:\nu_n}^{(n)}\}_n$ and consequently contradicts the relation (2.12). However, if such an $\varepsilon'_{i,1} > 0$ does not exist we have $\varlimsup_{n \rightarrow \infty} P(Z_{\hat{r}_i:[n\beta]}^{(n)} \geq x_i) = 0$, which in view of Lemma 2.1 (relation (2.10)) leads to the following chain of implications $(\forall x_i > 0) P(Z_{\hat{r}_i:[n\beta]}^{(n)} \geq x_i) \rightarrow 0 \Rightarrow \Gamma_{R_{r_i}}([N\beta]\bar{L}_m(x_{in})) \rightarrow 0 \Rightarrow [N\beta]\bar{L}_m(x_{in}) \rightarrow 0 \Rightarrow N\bar{L}_m(x_{in}) \rightarrow 0$ (since $N\bar{L}_m(x_{in}) \rightarrow 0$)

$\Rightarrow \Gamma_{R_{r_i}}(N\bar{L}_m(x_{in})) \rightarrow 0 \Rightarrow P(Z_{\hat{r}_i:n}^{(n)} \geq x_i) \rightarrow 0$, which contradicts the assumption (a). Consider the assumption (b). Since $H(z)$ is a df we can find a positive integer δ and real number $\alpha > 0$ such that

$$P\left(\frac{\nu_n}{n} \leq \delta\right) \geq \alpha, \quad \text{for sufficiently large } n. \quad (2.16)$$

Therefore, in view of (2.16) and the inequality (2.15), we have

$$\begin{aligned} P(Z_{\hat{r}_i:\nu_n}^{(n)} < x_i) &\geq \sum_{t=r}^{\delta n} P(Z_{\hat{r}_i:t}^{(n)} < x_i)P(\nu_n = t) \geq P(Z_{\hat{r}_i:\delta n}^{(n)} < x_i)P\left(\frac{\nu_n}{n} \leq \delta\right) \\ &\geq \alpha P(Z_{\hat{r}_i:\delta n}^{(n)} < x_i), \quad i = 1, 2. \end{aligned}$$

Hence, we get $\overline{\lim}_{n \rightarrow \infty} P(Z_{\hat{r}_i:\nu_n}^{(n)} < x_i) \geq \alpha \overline{\lim}_{n \rightarrow \infty} P(Z_{\hat{r}_i:\delta n}^{(n)} < x_i)$. By using Lemma 2.1 (relation (2.11)) and applying the same argument as in the case (a), it is easy to show that the last inequality leads to a contradiction (the last inequality, in view of the assumption (b)), which yields that the sequences $\{Z_{\hat{r}_i:n}^{(n)}\}_n, i = 1, 2$, is not stochastically bounded at the left. This completes the proof that the sequences $\{Z_{\hat{r}_i:n}^{(n)}\}_n, i = 1, 2$, are stochastically bounded. Now, if $\Phi_{\hat{r}_i:n}^{(m,k)}(x_{in}), i = 1, 2$, did not converge weakly, then we could select two subsequences $\{n'\}$ and $\{n''\}$ such that $\Phi_{\hat{r}_i:n'}^{(m,k)}(x_{in'})$ would converge weakly to $\hat{\Phi}_{r_i}'^{(m,k)}(x_i)$ and $\Phi_{\hat{r}_i:n''}^{(m,k)}(x_{in''})$ to another limit df $\hat{\Phi}_{r_i}''^{(m,k)}(x_i)$. In this case we get (by repeating the first part of Theorem 2.1 for the univariate case and for the two subsequences $\{n'\}, \{n''\}$)

$$\bar{\Omega}_{r_i}^{(m,k)}(x_i) = \int_0^\infty \left[1 - \Gamma_{R_{r_i}}(z\kappa_i'^{(m+1)})\right] dH(z) = \int_0^\infty \left[1 - \Gamma_{R_{r_i}}(z\kappa_i''^{(m+1)})\right] dH(z).$$

However, Lemma 3.2 in Barakat (1997) shows that the last equalities, cannot hold unless $\kappa_i' \equiv \kappa_i'', i = 1, 2$. Hence the relation (2.13) is proved. Hence, the proof of Theorem 2.1 is completed. ■

Let \mathcal{G} and \mathcal{G}_ν be the classes of all possible limit df's in (i) and (iii), respectively. The class \mathcal{G} is fully determined by Barakat et al. (2014b). Furthermore, let S and S_ν be the necessary and sufficient conditions for the validity of the relations (i) and (iii), respectively. The following corollary characterizes the class \mathcal{G}_ν .

Corollary 2.1. For every df $\hat{\Psi}^{(m,k)}(x, y)$ in \mathcal{G}_ν there exists a unique df $\hat{\Phi}^{(m,k)}(x, y)$ in \mathcal{G} , such that $\hat{\Psi}^{(m,k)}(x, y)$ is uniquely determined by the representation (iii). Moreover, $S = S_\nu$.

Proof of corollary 2.1. Let us first prove the implication $\{\hat{\Phi}_{r,s}'^{(m,k)}(x, y) \neq \hat{\Phi}_{r,s}''^{(m,k)}(x, y)\} \Rightarrow \{\hat{\Psi}_{r,s}'^{(m,k)}(x, y) \neq \hat{\Psi}_{r,s}''^{(m,k)}(x, y)\}$. If we assume the contrary, we get $\hat{\Psi}_{r,s}'^{(m,k)}(x, y) = \hat{\Psi}_{r,s}''^{(m,k)}(x, y)$, while $\hat{\Phi}_{r,s}'^{(m,k)}(x, y) \neq \hat{\Phi}_{r,s}''^{(m,k)}(x, y)$. Appealing to the first part of Theorem

2.1, we get $\int_0^\infty [1 - \Gamma_{R_{r_i}}(z\kappa_i'^{(m+1)})] dH(z) = \int_0^\infty [1 - \Gamma_{R_{r_i}}(z\kappa_i''^{(m+1)})] dH(z), i = 1, 2$. The last equalities, as we have seen before, from Lemma 3.2 in Barakat (1997), cannot hold unless $\kappa_i' = \kappa_i'', i = 1, 2$. Therefore, Corollary 2.1 is followed as a consequence of Theorem 2.1 and the last implication. This completes the proof of Corollary 2.1. ■

Theorem 2.2. Consider the following three conditions :

$$\Phi_{r,s;n}^{(m,k)}(x_n, y_n) = P(Z_{r,s;n}^{(n)} < \mathbf{x}) = P(Z_{r;n}^{(n)} < x, Z_{s;n}^{(n)} < y) \xrightarrow{\frac{w}{n}} \Phi_{r,s}^{(m,k)}(x, y), x \leq y, \quad (i)$$

$$H_n(nz) = P(\frac{\nu_n}{n} < z) \xrightarrow{\frac{w}{n}} H(z), \quad (ii)$$

$$\begin{aligned} \Phi_{r,s;\nu_n}^{(m,k)}(x_n, y_n) &= P(Z_{r,s;\nu_n}^{(n)} < \mathbf{x}) = P(Z_{r;\nu_n}^{(n)} < x, Z_{s;\nu_n}^{(n)} < y) \xrightarrow{\frac{w}{n}} \Psi_{r,s}^{(m,k)}(x, y) \\ &= \int_0^\infty \underline{\Omega}_{r,s}^{(m,k)}(z\rho_1, z\rho_2) dH(z). \end{aligned} \quad (iii)$$

Then any two of the above conditions imply the remaining one, where $x_n = c_n x + d_n, y_n = c_n y + d_n, c_n > 0, d_n$ are suitable normalizing constants, $1 \leq r < s \leq n, Z_{r;n}^{(n)} = \frac{X_{r;n} - d_n}{c_n}, i = 1, 2, \Phi_{r,s}^{(m,k)}(x, y)$ is a non-degenerate df, $H(z)$ is a df with $H(+0) = 0$,

$$\underline{\Omega}_{r,s}^{(m,k)}(\rho_1, \rho_2) = \begin{cases} \Gamma_s(\rho_2), & x \geq y, \\ \frac{1}{(r-1)!} \int_0^{\rho_1} \Gamma_{s-r}(\rho_2 - u) u^{r-1} e^{-u} du, & x \leq y, \end{cases}$$

$\rho_i = \mathcal{V}_{j;\beta}(x_i), i = 1, 2, j \in \{1, 2, 3\}, \mathcal{V}_{1;\beta}(x_i) = (-x_i)^{-\beta}, x_i \leq 0; \mathcal{V}_{2;\beta}(x_i) = x_i^\beta, x_i > 0, \beta > 0$ and $\mathcal{V}_3(x_i) = \mathcal{V}_{3;\beta}(x_i) = e^{x_i}, \forall x_i$.

Theorem 2.3. Consider the following three conditions :

$$\begin{aligned} \Phi_{r,\hat{s};n}^{(m,k)}(x_n, y_n) &= P(Z_{r,\hat{s};n}^{(n)} < \mathbf{x}) = P(Z_{r;n}^{(n)} < x, Z_{\hat{s};n}^{(n)} < y) \xrightarrow{\frac{w}{n}} \Phi_r^{(m,k)}(x) \hat{\Phi}_s^{(m,k)}(y) \\ &= \Gamma_r(\rho_1) [1 - \Gamma_{R_s}(\kappa_2^{m+1})], 1 \leq r, s \leq n, \end{aligned} \quad (i)$$

$$H_n(nz) = P(\frac{\nu_n}{n} < z) \xrightarrow{\frac{w}{n}} H(z), \quad (ii)$$

$$\Phi_{r,\hat{s};\nu_n}^{(m,k)}(x_n, y_n) = P(Z_{r,\hat{s};\nu_n}^{(n)} < \mathbf{x}) = P(Z_{r;\nu_n}^{(n)} < x, Z_{\hat{s};\nu_n}^{(n)} < y) \xrightarrow{\frac{w}{n}} \underline{\Omega}_r^{(m,k)}(z\rho_1) \overline{\Omega}_s^{(m,k)}(z\kappa_2). \quad (iii)$$

Then any two of the above conditions imply the remaining one, where $x_n = c_n x + d_n, y_n = a_n y + b_n, a_n, c_n > 0, b_n, d_n$ are suitable normalizing constants, $Z_{r;n}^{(n)} = \frac{X_{r;n} - d_n}{c_n}, Z_{\hat{s};n}^{(n)} = \frac{X_{\hat{s};n} - b_n}{a_n}, \Phi_r^{(m,k)}(x), \hat{\Phi}_s^{(m,k)}(y)$ are non-degenerate df's, $H(z)$ is a df with $H(+0) = 0$,

$$\underline{\Omega}_r^{(m,k)}(z\rho_1) = \int_0^\infty \Gamma_r(z\rho_1) dH(z) \text{ and } \overline{\Omega}_s^{(m,k)}(z\kappa_2) = \int_0^\infty [1 - \Gamma_{R_s}(z\kappa_2^{m+1})] dH(z).$$

Proof of Theorems 2.2 and 2.3. Without significant modifications, the method of the proof of Theorems 2.2 and 2.3 are the same as that Theorem 2.1, except only the obvious changes. Hence, for brevity the details of the proof are omitted. ■

2.2 The interrelation of ν_n and the basic rv's is not restricted

When the interrelation between the random index and the basic variables is not restricted, parallel theorem of Theorem 2.1 may be proved by replacing the condition (ii) by a stronger one. Namely, the weak convergence of the df $H_n(nz)$ must be replaced by the convergence in probability of the rv $\frac{\nu_n}{n}$ to a positive rv \mathcal{T} . However, the key ingredient of the proof of this parallel result is to prove the mixing property, due to Rényi (see, Barakat and Nigm, 1996) of the sequence of order statistics under consideration. In the sense of Rényi a sequence $\{u_n\}$ of rv's is called mixing if for any event \mathcal{E} of positive probability, the conditional df of $\{u_n\}$, under the condition \mathcal{E} , converges weakly to a non-degenerate df, which does not depend on \mathcal{E} , as $n \rightarrow \infty$. The following lemma proves the mixing property for the sequence $\{Z_{\hat{r},\hat{s}:n}^{(n)}\}_n$.

Lemma 2.2. Under the condition (i) in Theorem 2.1 the sequence $\{Z_{\hat{r},\hat{s}:n}^{(n)}\}_n$ is mixing.

Proof. The lemma will be proved if one shows the relation $P(Z_{\hat{r},\hat{s}:n}^{(n)} < \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} < \mathbf{x}) \xrightarrow{w} \hat{\Phi}_{r,s}^{(m,k)}(x, y)$, for all integers $l = r, r+1, \dots$. The sufficiency of the above relation can easily be proved as a direct multivariate extension of Lemma 6.2.1, of Galambos (1987). However, this relation is equivalent to

$$P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) \xrightarrow{w} \overline{\Phi}_{r,s}^{(m,k)}(x, y), \quad (2.17)$$

where $\overline{\Phi}_{r,s}^{(m,k)}(x, y)$ is the survival function of the limit df $\hat{\Phi}_{r,s}^{(m,k)}(x, y)$, i.e.,

$$\overline{\Phi}_{r,s}^{(m,k)}(x, y) = 1 - \hat{\Phi}_r^{(m,k)}(x) - \hat{\Phi}_s^{(m,k)}(y) + \hat{\Phi}_{r,s}^{(m,k)}(x, y).$$

Therefore, our lemma will be established if one proves the relation (2.17). Now, we can write

$$\begin{aligned} P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) &= P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(n)} < \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) \\ &+ P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(n)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}). \end{aligned} \quad (2.18)$$

Bearing in mind that all $X_{r:n}, r = 1, 2, \dots, n$, are i.i.d rv's, the first term in (2.18) can be written in the form

$$\begin{aligned} P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(n)} < \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) &= P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(n)} < \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) \\ &= P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}) - P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(n)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}), \end{aligned}$$

where

$$Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} = (Z_{\hat{r}:(n-l)}^{*(n)}, Z_{\hat{s}:(n-l)}^{*(n)}),$$

$$Z_{\hat{r}:(n-l)}^{*(n)} = ((r\text{th largest of } X_{1,l+1:n-l}, X_{1,l+2:n-l}, \dots, X_{1,n:n-l}) - b_n)/a_n,$$

$$Z_{\hat{s}:(n-l)}^{*(n)} = ((s\text{th largest of } X_{2,l+1:n-l}, X_{2,l+2:n-l}, \dots, X_{2,n:n-l}) - b_n)/a_n.$$

Therefore, in view of (2.18), we have

$$P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) = P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}) - \Delta_n(\mathbf{x}), \quad (2.19)$$

where $\Delta_n(\mathbf{x}) = P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) - P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}, Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x} \mid Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x})$. By using the well-known inequalities $Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \leq Z_{\hat{r},\hat{s}:n}^{(n)}$ and $P(B \cap C) - P(A \cap C) \leq P(B) - P(A)$, for any three events A, B and C , for which $A \subseteq B$, we get

$$0 \leq \Delta_n(\mathbf{x})P(Z_{\hat{r},\hat{s}:l}^{(l)} \geq \mathbf{x}) \leq P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x}) - P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}). \quad (2.20)$$

On the other hand, by virtue of the condition (i) in Theorem 2.1, it is easy to prove that

$$\lim_{n \rightarrow \infty} P(Z_{\hat{r},\hat{s}:(n-l)}^{*(n)} \geq \mathbf{x}) = \lim_{n \rightarrow \infty} P(Z_{\hat{r},\hat{s}:n}^{(n)} \geq \mathbf{x}) = \overline{\Phi}_{r,s}^{(m,k)}(x, y) \quad (2.21)$$

(note that $N\overline{L}_m(x_{in}) \xrightarrow[n]{} \kappa_i^{m+1} \Rightarrow (N-l)\overline{L}_m(x_{in}) \xrightarrow[n]{} \kappa_i^{m+1}$, $\forall x'_i s$ for which $\kappa_i < \infty$, $i = 1, 2$). By combining the relations (2.19)-(2.21), the proof of the relation (2.17) follows immediately. Hence the required result. ■

Considering the facts that the normalizing constants, which may be used in the bivariate extreme case are the same as those for the univariate case, and the limit $\text{df } \hat{\Phi}_{r,s}^{(m,k)}(x, y)$ is continuous, we can easily by using Lemma 2.2, show that the proof of the following theorem follows without any essential modifications as a direct multivariate extension of the proof of Theorem 2.1 in Barakat and El Shandidy (1990), except only the obvious changes.

Theorem 2.4. Consider the condition

$$\frac{\nu_n}{n} \xrightarrow[n]{p} \mathcal{T}, \quad (ii)'$$

where \mathcal{T} is a positive rv. Under the conditions of Theorem 2.1, we have the implication

$$(i) + (ii)' \Rightarrow (iii).$$

3 Illustrative examples

The range and midrange are widely used, particularly in statistical quality control as an estimator of the dispersion tendency and in setting confidence intervals for the population standard deviation as well as in Monte Carlo methods. In fact, the range itself is a very simple measure of dispersion, gives a quick and easy to estimate indication about the

spread of data. Let us defines the random generalized ranges $\mathcal{R}_{\nu_n}(m, k) = X_{\nu_n:\nu_n} - X_{1:\nu_n}$ and the random generalized midranges $\mathcal{V}_{\nu_n}(m, k) = \frac{X_{1:\nu_n} + X_{\nu_n:\nu_n}}{2}$, $X_{1:\nu_n} = X(1, \nu_n, m, k)$ and $X_{\nu_n:\nu_n} = X(\nu_n, \nu_n, m, k)$. The normalized generalized ranges and the normalized generalized midranges are defined by $\mathcal{R}_{\nu_n}^{(n)}(m, k) = A_{n:r}^{-1}(\mathcal{R}_{\nu_n}(m, k) - B_{n:r})$ and $\mathcal{V}_{\nu_n}^{(n)}(m, k) = A_{n:v}^{-1}(\mathcal{V}_{\nu_n}(m, k) - B_{n:v})$, respectively, where $A_{n:r} = 2A_{n:v} = a_n > 0$, $B_{n:r} = b_n - d_n$ and $B_{n:v} = \frac{1}{2}(b_n + d_n)$ are suitable normalizing constants. In this section, some illustrative examples for the most important distribution functions are obtained, which lend further support to our theoretical results. In the following examples we consider an important practical situation when ν_n has a geometric distribution with mean n . In this case we can easily show that $P(\frac{\nu_n}{n} < z) \xrightarrow{\frac{w}{n}} H(z) = 1 - e^{-z} (z \geq 0)$.

Example 3.1 (standard Cauchy distribution). Let $F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x$. Then $\alpha = \beta = 1$. In view of Theorems 1.1 and 2.1, Part 1, in Barakat et al. (2015b), we get, after some algebra,

$$\eta = \lim_{n \rightarrow \infty} \eta_n = \lim_{n \rightarrow \infty} \frac{a_n}{c_n} = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m > 0, \\ \infty, & \text{if } -1 < m < 0. \end{cases}$$

The random generalized ranges and midranges, for standard Cauchy distribution are given by, if $m = 0$,

$$P(\mathcal{R}_{\nu_n}^{(n)}(0, k) \leq r) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_0^\infty [1 - \Gamma_k(z(r - y)^{-1})] y^{-2} z e^{-z(1+y^{-1})} dy dz$$

and

$$P(\mathcal{V}_{\nu_n}^{(n)}(0, k) \leq v) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_{-\infty}^\infty [1 - \Gamma_k(z(v - y)^{-1})] y^{-2} z e^{-z(1-y^{-1})} dy dz,$$

respectively, with $A_{n:r} = 2A_{n:v} = a_n$, $B_{n:r} = B_{n:v} = 0$. Moreover, if $m > 0$,

$$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow{\frac{w}{n}} 1 - e^{-r^{-1}}$$

and

$$P(\mathcal{V}_{\nu_n}^{(n)}(m, k) \leq v) \xrightarrow{\frac{w}{n}} e^{-(-v)^{-1}},$$

respectively, with $A_{n:r} = 2A_{n:v} = c_n$, $B_{n:r} = B_{n:v} = 0$. Finally, when $-1 < m < 0$,

$$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow{\frac{w}{n}} \int_0^\infty [1 - \Gamma_\ell(zr^{-(m+1)})] e^{-z} dz$$

and the df of $\mathcal{V}_{\nu_n}^{(n)}(m, k)$ converge weakly to the same limit, with $A_{n:r} = 2A_{n:v} = a_n$, $B_{n:r} = B_{n:v} = 0$.

Example 3.2 (Pareto distribution). It can be shown that, for the Pareto distribution $F(x) = (1 - x^{-\sigma})I_{[1,\infty)}(x)$, $\sigma > 0$. Therefore, in view of Theorems 1.1 and 2.1, Part 1, in Barakat et al. (2015b), since $\frac{c_n}{a_n} \xrightarrow{n} 0, \forall m$. Then

$$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow{\frac{w}{n}} \int_0^\infty \left[1 - \Gamma_\ell(zr^{-\sigma(m+1)})\right] e^{-z} dz$$

and the df of $\mathcal{V}_{\nu_n}^{(n)}(m, k)$ converge weakly to the same limit, where $A_{n:r} = 2A_{n:v} = a_n, B_{n:r} = B_{n:v} = 0$.

Example 3.3 (uniform distribution). For the uniform $(-\theta, \theta)$ distribution, by using Theorem 2.1, Part 1, in Barakat et al. (2015b), since $\frac{a_n}{c_n} \xrightarrow{n} 1$, if $m = 0$. Then

$$P(R_{\nu_n}^{(n)}(0, k) \leq r) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_{-\infty}^0 [1 - \Gamma_k(z(y-r))] ze^{z(y-1)} dy dz$$

and

$$P(V_{\nu_n}^{(n)}(0, k) \leq v) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_{-\infty}^\infty [1 - \Gamma_k(z(y-v))] ze^{-z(y+1)} dy dz,$$

respectively, with $A_{n:r} = 2A_{n:v} = a_n, B_{n:r} = B_{n:v} = 0$.

Example 3.4 (Beta(α, β) distribution). For the beta distribution $F(x; \alpha, \beta), 0 \leq x \leq 1, \alpha, \beta > 0$. Therefore, in view of Theorem 2.2, Part 5, in Barakat et al. (2015b), if $\alpha = (m+1)\beta$. Then

$$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_{-\infty}^0 [1 - \Gamma_\ell(z(y-r)^\alpha)] z\eta\alpha(-\eta y)^{\alpha-1} e^{-z(1+(-\eta y)^\alpha)} dy dz$$

and

$$P(\mathcal{V}_{\nu_n}^{(n)}(m, k) \leq v) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_{-\infty}^\infty [1 - \Gamma_\ell(z(y-v)^\alpha)] z\eta\alpha(\eta y)^{\alpha-1} e^{-z(1+(\eta y)^\alpha)} dy dz,$$

respectively, where $\eta = (\frac{\beta}{c})^{\frac{1}{\beta}} (\frac{m+1}{c\alpha})^{\frac{1}{\alpha}}, c = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$. Clearly, the same result holds for the power distribution $F(x; \alpha, 1)$.

Example 3.5 (standard normal, logistic, Laplace, and log-normal distributions).

After some algebra, we get,

$$\eta^{-1} = \lim_{n \rightarrow \infty} \frac{c_n}{a_n} = \begin{cases} \frac{1}{\sqrt{m+1}}, & \text{for the normal distribution,} \\ 1, & \text{for the logistic and Laplace distribution,} \\ 0, & \text{for the log-normal distribution.} \end{cases}$$

Moreover, for the standard normal, logistic, and Laplace distributions, we get

$$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow{\frac{w}{n}} \int_0^\infty \int_{-\infty}^\infty [1 - \Gamma_\ell(ze^{(y-r)(m+1)})] z\eta e^{-\eta y} e^{-z(1+e^{-\eta y})} dy dz$$

and

$$P(\mathcal{V}_{\nu_n}^{(n)}(m, k) \leq v) \xrightarrow[n]{w} \int_0^\infty \int_{-\infty}^\infty [1 - \Gamma_\ell(ze^{(y-v)(m+1)})] z\eta e^{\eta y} e^{-z(1+e^{\eta y})} dy dz,$$

for the standard normal distribution ($m = 0, k = 1$). Then

$$P(\mathcal{R}_{\nu_n}(0, 1) \leq a_n r + 2b_n) \xrightarrow[n]{w} \int_0^\infty \frac{y^2 dy}{(y^2 + y + e^{-r})^2} = \begin{cases} f_1(r), & r < \ln 4, \\ \frac{2}{3}, & r = \ln 4, \\ f_2(r), & r > \ln 4, \end{cases}$$

where

$$\begin{aligned} f_1(r) &= (4e^{-r} - 1)^{-1} \left[\frac{4e^{-r}}{\sqrt{4e^{-r} - 1}} \cot^{-1} \left(\frac{1}{\sqrt{4e^{-r} - 1}} \right) - 1 \right], \\ f_2(r) &= (1 - 4e^{-r})^{-1} \left[1 - \frac{2e^{-r}}{\sqrt{1 - 4e^{-r}}} \ln \frac{1 + \sqrt{1 - 4e^{-r}}}{1 - \sqrt{1 - 4e^{-r}}} \right], \\ a_n &= \frac{1}{\sqrt{2 \ln n}}, \quad b_n = \sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}} \end{aligned}$$

and

$$P(\mathcal{V}_{\nu_n}(0, 1) \leq a_n v) \xrightarrow[n]{w} 1 - \int_0^\infty \frac{dy}{(y(e^{2v} + 1) + 1)^2} = (1 + e^{-2v})^{-1}.$$

Finally, for the log-normal distribution,

$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow[n]{w} \int_0^\infty [1 - \Gamma_\ell(ze^{-r(m+1)})] e^{-z} dz$ and the df of $\mathcal{V}_{\nu_n}^{(n)}(m, k)$ converge weakly to the same limit.

Example 3.6 (exponential (σ) and Rayleigh (σ) distributions). In view of Theorem 2.2, Part 4 in Barakat et al. (2015b). Then

$P(\mathcal{R}_{\nu_n}^{(n)}(m, k) \leq r) \xrightarrow[n]{w} \int_0^\infty [1 - \Gamma_\ell(ze^{-r(m+1)})] e^{-z} dz$ and the df of $\mathcal{V}_{\nu_n}^{(n)}(m, k)$ converge weakly to the same limit, for exponential (σ) and Rayleigh (σ) distributions.

Acknowledgements

Elsawah's work was partially supported by the UIC GRANT R201409 and the Zhuhai Premier Discipline Grant and Qin's work was partially supported by the National Natural Science Foundation of China (Nos. 11271147, 11471135, 11471136).

References

- [1] Barakat, H. M. (1997). Asymptotic properties of bivariate random extremes. J. Stat. Plann. Inference, 61, 203-217.
- [2] Barakat, H. M. (2007). Limit theory of generalized order statistics. J. Statist. Plann. Inference. Vol. 137, No. 1, 1-11.

- [3] Barakat, H. M. and El Shandidy, M. A. (1990). On the limit distribution of the extremes of a random number of independent random variables. *J. Statist. Plann. Inference*, 26, 353-361.
- [4] Barakat, H. M. and Nigm, E. M. (1996). The mixing property of order statistics with some applications. *Bull. Malaysian Math. Soc. (Second Series)* 19, 39-52.
- [5] Barakat, H. M., Nigm, E. M. and Abd Elgawad, M. A. (2014a). Limit theory for joint generalized order statistics, *REVSTAT Statistical Journal* 12(3), 199-220.
- [6] Barakat, H. M., Nigm, E. M. and Abd Elgawad, M. A. (2014b). Limit theory for bivariate extreme generalized order statistics and dual generalized order statistics, *ALEA, Lat. Am. J. Probab. Math. Stat.* 11 (1), 331-340.
- [7] Barakat, H. M., Nigm, E. M. and Al-Awady, M. A. (2015a). Asymptotic properties of multivariate order statistics with random index. *Bull. Malays. Math. Sci. Soc.* 38(1), 289-301.
- [8] Barakat, H. M., Nigm, E. M. and Elsayah, A. M. (2015b). Asymptotic distributions of the generalized range, midrange, extremal quotient, and extremal product, with a comparison study. *Comm. Statist. Theory Methods*, 44, 900-913.
- [9] Cramer, E. (2003). Contributions to Generalized Order Statistics. Habilitationsschrift, Reprint, University of Oldenburg.
- [10] Feller, W. (1979). An introduction to probability theory and its applications. Vol. 2, John Wiley & Sons. Inc. (Wiley Eastern University edition).
- [11] Galambos, J. (1978, 1987). The asymptotic theory of extreme order statistics, New York, Wiley, (1st ed.) Kreiger. FI (2nd ed.).
- [12] Kamps, U. (1995). A Concept of Generalized Order Statistics. Teubner, Stuttgart.
- [13] Nasri-Roudsari, D. (1996). Extreme value theory of generalized order statistics, *J. Statist. Plann. Inference*, 55, 281-297.
- [14] Smirnov, N.V. (1952). Limit distribution for terms of a variational series. *Trans. Amer. Math. Soc. Ser. I.* 11, 82-143.